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# ***Stability and Unstability Matrices for Linear Evolution Variational Inequalities***

Daniel Goeleven — Bernard Brogliato

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## Stability and Unstability Matrices for Linear Evolution Variational Inequalities

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**Abstract:** This paper deals with the characterisation of the stability and unstability matrices for a class of unilaterally constrained dynamical systems, represented as linear evolution variational inequalities (LEVI). Such systems can also be seen as a sort of differential inclusion, or (in special cases) as linear complementarity systems, which in turn are a class of hybrid dynamical systems. Examples show that the stability of the unconstrained system and that of the constrained system, may drastically differ. Various criteria are proposed to characterize the stability or the instability of LEVI.

**Key-words:** Lyapunov stability, unilateral constraints, variational inequalities, convex analysis, hybrid dynamics, copositive matrices, stability matrices, unstability matrices.

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## Matrices stables et instables pour les inégalités variationnelles linéaires dynamiques

**Résumé :** Ce papier concerne la caractérisation de la stabilité au sens de Lyapunov, des inégalités variationnelles linéaires dynamiques. De tels systèmes peuvent aussi être vus comme des cas particuliers de systèmes de complémentarité, ou des inclusions différentielles. Des exemples illustrent les différences entre la stabilité des systèmes sans contraintes, et celle des systèmes avec contraintes unilatérales.

**Mots-clés :** Stabilité de Lyapunov, contraintes unilatérales, inégalités variationnelles, analyse convexe, dynamique hybride, matrices copositives, matrices stables, matrices instables.

# 1 Introduction

The stability of stationary solutions of dynamic systems constitutes a very important topic in Applied Mathematics and Engineering. It is well-known that in the case of a large class of nonlinear differential equations the spectrum of "linearized" operators determines the Lyapunov stability of an equilibrium. However, many important problems in engineering (see [15] [21] [22] [23] [26]) involve inequalities in their mathematical formulation and consequently possess intrinsic nonsmoothness. For these last cases the question of stability is much more complicated to be investigated, as it is the case in general for hybrid dynamical systems, see e.g. [31] [32] [33] [7] [8]. An interesting class of unilaterally constrained dynamical systems can be represented under the formalism of variational inequalities, see e.g. [19]. Variational inequalities are widely used in applied mathematics and various fields of science with applications to behaviour of oligopolistic markets, urban transportation networks, traffic networks, international trade, agricultural and energy markets (spatial price equilibria) [34] [35] [36]. The stability of variational inequalities has been investigated by various authors, see e.g. [9] [10] [11] [6] [37]. In this paper we will deal with the following class of dynamical systems:

Let  $K \subset \mathbb{R}^n$  be a nonempty closed convex set. Let  $A \in \mathbb{R}^{n \times n}$  be a given matrix and  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  a nonlinear operator. For  $(t_0, x_0) \in \mathbb{R} \times K$ , we consider the problem  $P(t_0, x_0)$ : Find a function  $t \rightarrow x(t)$  ( $t \geq t_0$ ) with  $x \in C^0([t_0, +\infty); \mathbb{R}^n)$ ,  $\frac{dx}{dt} \in L_{loc}^\infty(t_0, +\infty; \mathbb{R}^n)$  and such that:

$$\begin{cases} x(t) \in K, \ t \geq t_0 \\ \langle \frac{dx}{dt}(t) + Ax(t) + F(x(t)), v - x(t) \rangle \geq 0, \ \forall v \in K, \text{ a.e. } t \geq t_0 \\ x(t_0) = x_0 \end{cases} \quad (1)$$

Here  $\langle \cdot, \cdot \rangle$  denotes the euclidean scalar product in  $\mathbb{R}^n$ . The corresponding norm is denoted by  $\| \cdot \|$ . The system in (1) is an evolution variational inequality which we denote as  $\text{LEVI}(A, K)$  when  $F \equiv 0$ . It follows from standard convex analysis that (1) can be rewritten equivalently as the differential inclusion

$$\frac{dx}{dt}(t) + Ax(t) + F(x(t)) \in -N_K(x(t)) \quad (2)$$

where  $N_K(x(t)) = \{s \in \mathbb{R}^n : \langle s, v - x(t) \rangle \leq 0, \ \forall v \in K\}$  is the normal cone to  $K$  at  $x(t)$  [18]. In case  $K = \{x \in \mathbb{R}^n : Cx + d \geq 0\}$  for some matrix  $C \in \mathbb{R}^{m \times n}$  and vector  $d \in \mathbb{R}^m$ , we can rewrite (1) as

$$\begin{cases} \frac{dx}{dt}(t) + Ax(t) + F(x(t)) = C^T \lambda \\ 0 \leq y = Cx + d \perp \lambda \geq 0 \end{cases} \quad (3)$$

where  $\lambda \in \mathbb{R}^m$  is a Lagrange multiplier, and the second line of (3) means that both  $y$  and  $\lambda$  have to be non-negative, and orthogonal. System (3) belongs to the class of Linear Complementarity Systems (LCS) with relative degree between  $y$  and  $\lambda \geq 1$  [17] [15] [16]. Variational inequalities and complementarity are known to be closely related [29] [30] [34].

### Remark 1

- Mechanical systems with unilateral constraints do not belong to the class of systems studied in this paper. Indeed in (3) one has  $y = Dx + d$  with  $D \neq C$  for mechanical systems.
- The problems related to positively invariant sets and control holdability of sets (see e.g. [27] [1] [2]) are essentially different from what is studied in this paper. Indeed (see theorem 1 below) trajectories of (1) remain in  $K$  for all  $A$  and all  $t \geq t_0$  because the dynamics is modified on the boundary of  $K$ .
- Assume that  $K = \{x \in \mathbb{R}^n : g(x) \geq 0\}$  for some convex function  $g(\cdot)$ . Then  $\partial K$  may possess an infinity of corners. The framework in (1) encompasses such cases.

The paper is organised as follows: in section 2 some theoretical results and definitions are presented. In section 3 matrices  $A$  in (1) that yield stable systems are studied. Section 4 is devoted to characterize unstable matrices  $A$ . Section 5 concerns stable and unstable LEVI( $A, K$ ). Section 6 shows that stability is preserved under small nonlinear disturbances. Section 7 applies the results to some concrete problems. Conclusions end the paper.

## 2 Abstract Results

Let us first specify some conditions ensuring the existence and uniqueness of the initial value problem  $P(t_0, x_0)$ . The following existence and uniqueness result is a direct consequence of [6, corollary 2.2].

**Theorem 1** *Let  $K$  be a nonempty closed convex subset of  $\mathbb{R}^n$  and let  $A \in \mathbb{R}^{n \times n}$  be a real matrix of order  $n$ . Suppose that  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  can be written as*

$$F = F_1 + \Phi'$$

*where  $F_1$  is Lipschitz continuous and  $\Phi \in C^1(\mathbb{R}^n; \mathbb{R})$  is convex. Let  $t_0 \in \mathbb{R}$  and  $x_0 \in K$  be given. Then there exists a unique  $x \in C^0([t_0, +\infty); \mathbb{R}^n)$  such that*

$$\frac{dx}{dt} \in L_{\text{loc}}^\infty([t_0, +\infty); \mathbb{R}^n) \quad (4)$$

$$x \text{ is right-differentiable on } [t_0, +\infty) \quad (5)$$

$$x(t_0) = x_0 \quad (6)$$

$$x(t) \in K, \quad t \geq t_0 \quad (7)$$

$$\left\langle \frac{dx}{dt}(t) + Ax(t) + F(x(t)), v - x(t) \right\rangle \geq 0, \quad \forall v \in K, \quad a.e. \quad t \geq t_0 \quad (8)$$

■

Suppose that the assumptions of Theorem 1 are satisfied and denote by  $x(\cdot; t_0, x_0)$  the unique solution of Problem  $P(t_0, x_0)$ . Suppose now in addition that  $0 \in K$  and

$$-F(0) \in N_K(0), \quad (9)$$

that is

$$\langle F(0), h \rangle \geq 0, \quad \forall h \in K.$$

Then

$$x(t; t_0, 0) = 0, \quad t \geq t_0,$$

i.e. the trivial solution 0 is the unique solution of problem  $P(t_0, 0)$ .

**Remark 2** The wellposedness results of Theorem 1 continue to hold for a controlled LEVI( $A, B, K$ ) defined as  $\langle \frac{dx}{dt}(t) + Ax(t) + Bu(t) + F(x(t)), v - x(t) \rangle \geq 0, \quad \forall v \in K$ , with  $B \in \mathbb{R}^{m \times n}$ ,  $u \in C^0([t_0, +\infty); \mathbb{R}^m)$  and  $\frac{du}{dt} \in L^1_{loc}([t_0, +\infty); \mathbb{R}^m)$ , see [6]. This is important in view of controllability issues. In relationship with (2) and (3), let us notice that if  $y = Cx + d + Bu(t)$  in (3), then the right-hand-side of (2) becomes equal to  $N_{K(u(t))}(x(t))$ , i.e. the convex set  $K(u(t)) = \{x \in \mathbb{R}^n : Cx + d + Bu(t) \geq 0\}$  is time varying. If  $A = 0$  and  $F(\cdot) = 0$  then the obtained system fits within Moreau's sweeping process [25]. This case is not studied in this paper. However some electrical circuits possess a relative degree 0 between  $y$  and  $u$  [16], so the extension of the presented results towards convex sets  $K(t)$  is valuable.

We may now define as in [6] the stability of the trivial solution. The stationary solution 0 is called stable if small perturbations of the initial condition  $x(t_0) = 0$  lead to solutions which remain in the neighborhood of 0 for all  $t \geq t_0$ , precisely:

**Definition 1** *The equilibrium point  $x = 0$  is said to be stable in the sense of Lyapunov if for every  $\varepsilon > 0$  there exists  $\eta = \eta(\varepsilon) > 0$  such that for any  $x_0 \in K$  with  $\|x_0\| \leq \eta$  the solution  $x(\cdot; t_0, x_0)$  of problem  $P(t_0, x_0)$  satisfies  $\|x(t; t_0, x_0)\| \leq \varepsilon, \quad \forall t \geq t_0$ .*

If in addition the trajectories of the perturbed solutions are attracted by 0 then we say that the stationary solution is asymptotically stable, precisely:



**Definition 2** *The equilibrium point  $x = 0$  is asymptotically stable if it is stable and there exists  $\delta > 0$  such that for any  $x_0 \in K$  with  $\|x_0\| \leq \delta$  the solution  $x(\cdot; t_0, x_0)$  of problem  $P(t_0, x_0)$  fulfills*

$$\lim_{t \rightarrow +\infty} \|x(t; t_0, x_0)\| = 0.$$

The notion of unstability is now given.

**Definition 3** *The equilibrium point  $x = 0$  is unstable if it is not stable, i.e. there exists  $\varepsilon > 0$  such that for any  $\eta > 0$ , one may find  $x_0 \in K$  with  $\|x_0\| \leq \eta$  and  $\bar{t} \geq t_0$  such that the solution  $x(\cdot; t_0, x_0)$  of problem  $P(t_0, x_0)$  verifies*

$$\|x(\bar{t}; t_0, x_0)\| > \varepsilon.$$

Let us now give general abstract theorems of stability, asymptotic stability and unstability in terms of generalized Lyapunov functions  $V \in C^1(\mathbb{R}^n; \mathbb{R})$ . The following results are particular cases of the ones proved in [6].

**Theorem 2** *Suppose that the assumptions of Theorem 1 together with condition (9) hold. Suppose that there exists  $\sigma > 0$  and  $V \in C^1(\mathbb{R}^n; \mathbb{R})$  such that*

(1)

$$V(x) \geq a(\|x\|), \quad x \in K, \quad \|x\| \leq \sigma,$$

with  $a : [0, \sigma] \rightarrow \mathbb{R}$  satisfying  $a(t) > 0, \forall t \in (0, \sigma)$ ;

(2)  $V(0) = 0$ ;

(3)  $x - V'(x) \in K, \quad x \in \partial K, \quad \|x\| \leq \sigma$

(4)  $\langle Ax + F(x), V'(x) \rangle \geq 0, \quad x \in K, \quad \|x\| \leq \sigma.$

Then the trivial solution of (7)-(8) is stable. ■

**Theorem 3** *Suppose that the assumptions of Theorem 1 together with condition (9) hold. Suppose that there exists  $\lambda > 0, \sigma > 0$  and  $V \in C^1(\mathbb{R}^n; \mathbb{R})$  such that*

(1)

$$V(x) \geq a(\|x\|), \quad \text{for all } x \in K, \quad \|x\| \leq \sigma,$$

with  $a : [0, \sigma] \rightarrow \mathbb{R}$  satisfying  $a(t) \geq ct^\tau, \forall t \in [0, \sigma],$  for some constants  $c > 0, \tau > 0$ ;

(2)  $V(0) = 0$ ;

(3)  $x - V'(x) \in K, \quad \text{for all } x \in \partial K, \quad \|x\| \leq \sigma$

$$(4) \quad \langle Ax + F(x), V'(x) \rangle \geq \lambda V(x), \text{ for all } x \in K, \|x\| \leq \sigma.$$

Then the trivial solution of (7)-(8) is asymptotically stable.  $\blacksquare$

Let us note that condition (3) implies  $-V'(x) \in T_K(x)$  for all  $x \in \partial K$ ,  $\|x\| \leq \sigma$ , where  $T_K(x)$  is the tangent cone to  $K$  at  $x$ , and  $V'(x)$  denotes the gradient of  $V(\cdot)$  at  $x$  [18, proposition 5.2.1].

We formulate now an unstability result.

**Theorem 4** *Suppose that the assumptions of Theorem 1 together with condition (9) hold. Suppose also that  $K \setminus \{0\} \neq \emptyset$ . If there exists  $V \in C^1(\mathbb{R}^n; \mathbb{R})$  and  $\alpha > 0$  such that*

(1)

$$V(x) \leq b(\|x\|), \quad x \in K,$$

with  $b : [0, +\infty) \rightarrow \mathbb{R}$  satisfying  $b(t) \leq kt^s$ ,  $\forall t \geq 0$ , for some constants  $k > 0$ ,  $s > 0$ ;

$$(2) \quad V(x) > 0, \quad x \in K, \quad x \neq 0 \text{ near } 0;$$

$$(3) \quad V'(x) \in K_\infty, \quad x \in \partial K;$$

$$(4) \quad \langle Ax + F(x), V'(x) \rangle \leq -\alpha V(x), \quad x \in K,$$

then the trivial solution (7)-(8) is unstable.  $\blacksquare$

The recession cone  $K_\infty$  will be defined in the next section.

### 3 Stability matrices on a closed convex set

The aim of this section is to introduce and study two classes of stability matrices. The results obtained here will be used later in this paper to construct generalized Lyapunov functions needed to apply the abstract stability results given in section 2.

Let us first recall some basic tools from convex analysis [18]. Let  $C$  be a nonempty subset of  $\mathbb{R}^n$ . We say that  $C$  is a cone if  $\lambda C \subset C, \forall \lambda > 0$ . A nonempty closed convex cone is characterized by the relations:

$$\lambda C \subset C, \forall \lambda > 0; \quad C + C \subset C; \quad 0 \in C.$$

Let  $D \subset \mathbb{R}^n$  be a nonempty closed convex set. The recession (or asymptotic) cone  $D_\infty$  of  $D$  is defined by

$$D_\infty = \bigcap_{\lambda > 0} \lambda(D - x_0),$$

where  $x_0$  is an arbitrary fixed element of  $D$ . In other words the recession cone of  $D$  is the set of directions from which one can go straight from any point  $x_0 \in D$  to infinity, while staying in  $D$  [18, §A 2.2]. The set  $D_\infty$  is a closed convex cone. One has

$$D + D_\infty \subset D$$

and thus, if  $0 \in D$  then  $D_\infty \subset D$ . If  $D$  is symmetric, i.e.  $x \in D \Rightarrow -x \in D$  then clearly  $D_\infty$  is a subspace of  $\mathbb{R}^n$ . If  $D$  is bounded then  $D_\infty = \{0\}$ . For  $x \in \mathbb{R}^n$  we denote by  $P_D(x)$  the unique solution of the variational inequality: Find  $z \in D$  such that

$$\langle z - x, y - z \rangle \geq 0, \forall y \in D.$$

The mapping  $P_D : \mathbb{R}^n \rightarrow D; x \mapsto P_D(x)$  is called the projection on  $D$ . If  $D$  is a subspace of  $\mathbb{R}^n$  then  $P_D$  is linear and symmetric. In this case, we write  $P_D(x) = P_D x$  where  $P_D \in \mathbb{R}^{n \times n}$  denotes the symmetric matrix of orthogonal projection on  $D$ .

The unit sphere in  $\mathbb{R}^n$  is here denoted by  $S_1$ , i.e.  $S_1 := \{x \in \mathbb{R}^n : \|x\| = 1\}$ . In the sequel  $A \in \mathbb{R}^{n \times n}$  denotes a real matrix of order  $n$  and  $K \subset \mathbb{R}^{n \times n}$  is a subset of  $\mathbb{R}^n$ . Let us also point out the following notations:  $\sigma(A)$  denotes the spectrum of  $A$ ,  $\rho(A)$  is the spectral radius of  $A$  and for  $\lambda \in \sigma(A)$ ,  $E_A(\lambda)$  denotes the corresponding eigenspace. The transpose of  $A$  is denoted by  $A^T$ ,  $\ker\{A\}$  is the null space of  $A$  and  $\text{tr}\{A\}$  is the trace of  $A$ . The interior and boundary of  $K$  are respectively denoted by  $\overset{\circ}{K}$  and  $\partial K$ . We assume that

(H<sub>1</sub>)  $K$  is closed

(H<sub>2</sub>)  $K$  is convex

(H<sub>3</sub>)  $0 \in K$ .

Only the possible additional conditions needed on  $K$  will be specified in this paper.

**Definition 4** *The matrix  $A \in \mathbb{R}^{n \times n}$  is Lyapunov positive semi-stable on  $K$  if there exists a matrix  $G \in \mathbb{R}^{n \times n}$  such that*

- (1)  $\inf_{x \in K \cap S_1} \langle Gx, x \rangle > 0$ ,
- (2)  $\langle Ax, [G + G^T]x \rangle \geq 0, \forall x \in K$ ,
- (3)  $x \in \partial K \Rightarrow [I - [G + G^T]]x \in K$ .

**Definition 5** *The matrix  $A \in \mathbb{R}^{n \times n}$  is Lyapunov positive stable on  $K$  if there exists a matrix  $G \in \mathbb{R}^{n \times n}$  such that*

- (1)  $\inf_{x \in K \cap S_1} \langle Gx, x \rangle > 0$ ,
- (2)  $\inf_{x \in K \cap S_1} \langle Ax, [G + G^T]x \rangle > 0$ ,
- (3)  $x \in \partial K \Rightarrow [I - [G + G^T]]x \in K$ .

**Remark 3** Condition (1) of definition 4 (and 5) is equivalent to the existence of a constant  $c > 0$  such that

$$\langle Gx, x \rangle \geq c \|x\|^2, \forall x \in K.$$

Recall that a matrix  $G \in \mathbb{R}^{n \times n}$  is said to be copositive on  $K$  if

$$\langle Gx, x \rangle \geq 0, \forall x \in K.$$

A matrix  $G \in \mathbb{R}^{n \times n}$  is said to be strictly copositive on  $K$  if

$$\langle Gx, x \rangle > 0, \forall x \in K \setminus \{0\}.$$

These classes of matrices play an important role in complementarity theory (see e.g. [20]). The set of copositive matrices strictly contains that of positive semi definite matrices [20, p.174].

Condition (3) of definitions 4 and 5 correspond to condition (3) in theorems 2 and 3. Let us here denote by  $\mathcal{P}_K$  (resp.  $\mathcal{P}_K^+$ ) the set of copositive (resp. strictly copositive) matrices on  $K$ . Let us also denote by  $\mathcal{P}_K^{++}$  the set of matrices satisfying condition (1) of definition 4, that is

$$\mathcal{P}_K^{++} = \{B \in \mathbb{R}^{n \times n} : \inf_{x \in K \cap S_1} \langle Bx, x \rangle > 0\}.$$

It is clear that

$$\begin{aligned} \mathcal{P}_K^{++} &\subset \mathcal{P}_K^+ \subset \mathcal{P}_K. \\ K_1 \subset K_2 &\Rightarrow \mathcal{P}_{K_2}^{++} \subset \mathcal{P}_{K_1}^{++} \end{aligned}$$

**Proposition 1** *If  $K$  is a cone then*

$$\mathcal{P}_K^{++} = \mathcal{P}_K^+$$

**Proof:** We know that  $\mathcal{P}_K^{++} \subset \mathcal{P}_K^+$ . It suffices to check that  $\mathcal{P}_K^+ \subset \mathcal{P}_K^{++}$ . Let  $B \in \mathcal{P}_K^+$ , that is  $\langle Bx, x \rangle > 0, \forall x \in K \setminus \{0\}$  and let us verify that there exists  $c > 0$  such that  $\langle Bx, x \rangle \geq c \|x\|^2, \forall x \in K \setminus \{0\}$ . If we suppose the contrary then we can find a sequence  $\{x_n\} \subset K \setminus \{0\}$  such that  $\langle Bx_n, x_n \rangle < \frac{1}{n} \|x_n\|^2$ . Let  $z_n := \frac{x_n}{\|x_n\|}$ . We have  $\|z_n\| = 1$  and thus there exists a subsequence  $\{z_{n_k}\}$  such that  $z_{n_k} \rightarrow z, \|z\| = 1$  and  $\langle Bz, z \rangle = \lim_{n_k \rightarrow \infty} \langle Bz_{n_k}, z_{n_k} \rangle \leq 0$ . This contradicts the strict copositivity of  $B$ . ■

**Remark 4** Let  $A \in \mathbb{R}^{n \times n}$  be a positive stable matrix, i.e.

$$re(\lambda) > 0, \forall \lambda \in \sigma(A). \quad (3.1)$$

Then there exists a matrix  $G \in \mathbb{R}^{n \times n}$  satisfying conditions (1) and (2) of Definition 5. Indeed, let  $H \in \mathbb{R}^{n \times n}$  be any positive definite matrix. From Lyapunov's theorem (see [24]) there exists a positive definite matrix  $G \in \mathbb{R}^{n \times n}$  satisfying the Lyapunov equation  $GA + A^T G = H$ . Thus  $\langle Ax, [G + G^T]x \rangle = \langle Hx, x \rangle > 0, \forall x \in \mathbb{R}^n \setminus \{0\}$ .

**Remark 5** A first step in finding a matrix  $G$  satisfying condition (2) of Definition 4 (resp. Definition 5) may consist to deal with the Lyapunov equation  $GA + A^T G = H$  and by taking  $H$  as a matrix of  $\mathcal{P}_K$  (resp.  $\mathcal{P}_K^{++}$ ). The Lyapunov equation can be written as

$$(A^T \otimes I + I \otimes A^T)g = h, \quad (3.3)$$

where  $g$  and  $h$  denotes the column  $n^2$  vectors formed from the rows of  $G$  and  $H$ , respectively, taken in order. Note also that by choosing  $G$  and  $H$  symmetric then, redundant equations and variables can be removed to give a system of order  $\frac{1}{2}n(n+1)$ .

Let us now denote by  $\mathcal{L}_K$  the set of Lyapunov positive semi-stable matrices on  $K$  and by  $\mathcal{L}_K^{++}$  the set of Lyapunov positive stable matrices on  $K$ . We see that

$$\begin{aligned} \mathcal{L}_K = \{ & A \in \mathbb{R}^{n \times n} : \exists G \in \mathcal{P}_K^{++} \text{ such that } (I - [G + G^T])(\partial K) \subset K \\ & \text{and } GA + A^T G \in \mathcal{P}_K \}. \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_K^{++} = \{ & A \in \mathbb{R}^{n \times n} : \exists G \in \mathcal{P}_K^{++} \text{ such that } (I - [G + G^T])(\partial K) \subset K \\ & \text{and } GA + A^T G \in \mathcal{P}_K^{++} \}. \end{aligned}$$

Let us note that  $G$  needs not be symmetric. Important classes of matrices satisfying the conditions of definition 4 and definition 5 are now given.

**Proposition 2** Suppose that  $A \in \mathcal{P}_K$ , i.e

$$\langle Ax, x \rangle \geq 0, \forall x \in K$$

Then  $A \in \mathcal{L}_K$ .

**Proof:** Let  $G = \frac{1}{2}I$ . Conditions (1) and (2) of definition 4 are clearly satisfied. Moreover  $(I - [G + G^T])x = 0 \in K$ . ■

**Proposition 3** Suppose that  $A \in \mathcal{P}_K^{++}$ , i.e.

$$\inf_{x \in K \cap S_1} \langle Ax, x \rangle > 0.$$

Then  $A \in \mathcal{L}_K^{++}$ .

**Proof:** As in the proof of Proposition 2, we see that the choice  $G = \frac{1}{2}I$  is convenient. ■

**Remark 6** From Propositions 2 and 3, we deduce that

$$\mathcal{P}_K \subset \mathcal{L}_K, \quad \mathcal{P}_K^{++} \subset \mathcal{L}_K^{++}.$$

We see now that, with some additional conditions imposed on the set  $K$ , it is possible to consider larger classes of matrices.

**Proposition 4** Suppose that  $K$  is a cone such that

$$x \in \partial K \Rightarrow x_i \bar{e}_i \in K \quad (i = 1, \dots, n)$$

where  $\bar{e}_i$  denotes the  $i$ -th canonical vector of  $\mathbb{R}^n$ . If there exists a positive diagonal matrix  $D$  such that

$$DA \in \mathcal{P}_K$$

then  $A \in \mathcal{L}_K$ .

**Proof:** We set  $G := \frac{1}{2\alpha}D$ , where  $\alpha := \max_{1 \leq i \leq n} \{d_{ii}\}$ . The matrix  $G$  is symmetric and positive definite. Moreover,  $\langle Ax, [G + G^T]x \rangle = \frac{1}{\alpha} \langle DAx, x \rangle$  and thus  $\langle Ax, [G + G^T]x \rangle \geq 0, \forall x \in K$ . Finally,  $(I - [G + G^T])x = \text{diag}_{1 \leq i \leq n} \{(1 - \frac{d_{ii}}{\alpha})x_i\} = \sum_{i=1}^n (1 - \frac{d_{ii}}{\alpha})x_i \bar{e}_i$ . Then using the assumptions on  $K$ , we see that for  $x \in \partial K$  we have  $(I - [G + G^T])x \in K$ . ■

**Proposition 5** Suppose that  $K$  is a cone such that

$$x \in \partial K \Rightarrow x_i \bar{e}_i \in K \quad (i = 1, \dots, n).$$

If there exists a positive diagonal matrix  $D$  such that

$$DA \in \mathcal{P}_K^{++}$$

then  $A \in \mathcal{L}_K^{++}$ .

**Proof:** As in Proposition 4 we set  $G := \frac{1}{2\alpha}D$  with  $\alpha := \max_{1 \leq i \leq n} \{d_{ii}\}$ . Here  $\langle Ax, [G + G^T]x \rangle = \frac{1}{\alpha} \langle DAx, x \rangle$  and thus  $\inf_{x \in K \cap S_1} \langle Ax, [G + G^T]x \rangle > 0$ . The proof is achieved as in Proposition 4. ■

**Remark 7** If  $A$  is a nonsingular  $M$ -matrix, i.e.  $A = sI - P; s > \rho(P)$ ,  $P$  is such that  $p_{ij} \geq 0$ , for all  $i, j \in \{1, \dots, n\}$ , then there exists a positive diagonal matrix  $D$  such that

$$H := DA + A^T D$$

is positive definite (see e.g. [12]). It results that  $\langle Ax, Dx \rangle = \frac{1}{2} \langle Ax, [D + D^T]x \rangle = \frac{1}{2} \langle [A^T D + DA]x, x \rangle = \frac{1}{2} \langle Hx, x \rangle \geq \frac{1}{2} \lambda_{\min}(\frac{H+H^T}{2}) \|x\|^2$ . Thus  $\inf_{x \in K \cap S_1} \langle DAx, x \rangle > 0$  and the condition required in Proposition 5 on the matrix  $A$  is satisfied.

**Proposition 6** *Suppose that  $K$  satisfies the property*

$$x \in \partial K \Rightarrow x_i \bar{e}_i \in K \ (i = 1, \dots, n).$$

*If there exists a positive diagonal matrix  $D$  such that*

$$(1) \quad \frac{\text{tr}(D)}{n-1} \geq \max_{1 \leq i \leq n} \{d_{ii}\}$$

$$(2) \quad DA \in \mathcal{P}_K.$$

*Then  $A \in \mathcal{L}_K$ .*

**Proof:** Let  $\alpha := \frac{\text{tr}(D)}{n-1}$ . We set  $G := \frac{1}{2\alpha}D$ . It is clear that  $G \in P_K^{++}$ . Moreover  $\langle Ax, [G + G^T]x \rangle = \frac{1}{\alpha} \langle DAx, x \rangle \geq 0$ . Finally  $(I - [G + G^T])x = \sum_{i=1}^n (1 - \frac{d_{ii}}{\alpha}) x_i \bar{e}_i$ . If  $x \in \partial K$  then  $x_i \bar{e}_i \in K$ . Moreover  $0 \leq \frac{d_{ii}}{\alpha} \leq 1$  since (1) holds and  $\sum_{i=1}^n (1 - \frac{d_{ii}}{\alpha}) = n - \frac{\text{tr}D}{\alpha} = 1$ . It results that for  $x \in \partial K$ , we have  $(I - [G + G^T])x \in K$ . ■

**Proposition 7** *Suppose that  $K$  satisfies the property*

$$x \in \partial K \Rightarrow x_i \bar{e}_i \in K \ (i = 1, \dots, n).$$

*If there exists a positive diagonal matrix  $D$  such that*

$$(1) \quad \frac{\text{tr}(D)}{n-1} \geq \max_{1 \leq i \leq n} \{d_{ii}\}$$

$$(2) \quad DA \in \mathcal{P}_K^{++}.$$

*Then  $A \in \mathcal{L}_K^{++}$ .*

**Proof:** With the choice of  $G$  as in the proof of Proposition 6, we see that conditions (1) and (3) of definition 5 hold. Moreover, as above, we check that  $\langle Ax, [G + G^T]x \rangle = \frac{1}{\alpha} \langle DAx, x \rangle$  and thus using assumption (2), we obtain  $\inf_{x \in K \cap S_1} \langle Ax, [G + G^T]x \rangle > 0$ . ■

**Proposition 8** *Suppose that  $K$  is symmetric. If there exists  $\lambda \geq 0$  such that*

$$(I + \lambda P_{K_\infty})A \in \mathcal{P}_K$$

*then  $A \in \mathcal{L}_K$ .*

**Proof:** We know that

$$\langle P_{K_\infty} x - x, y - P_{K_\infty} x \rangle \geq 0, \forall y \in K_\infty$$

and thus letting  $y = 0$  we get

$$\langle P_{K_\infty} x, x \rangle \geq \|P_{K_\infty} x\|^2.$$

Let

$$G = \frac{1}{2}[I + \lambda P_{K_\infty}].$$

Then

$$\langle Gx, x \rangle \geq \frac{1}{2} \|x\|^2 + \frac{\lambda}{2} \|P_{K_\infty}\|^2$$

and  $G \in \mathcal{P}_K^{++}$ . We have also  $\langle Ax, [G + G^T]x \rangle = \langle Ax, x \rangle + \lambda \langle Ax, P_{K_\infty}x \rangle = \langle Ax, x \rangle + \lambda \langle P_{K_\infty}Ax, x \rangle = \langle [I + \lambda P_{K_\infty}]Ax, x \rangle$ . Thus, by assumption  $\langle Ax, [G + G^T]x \rangle \geq 0, \forall x \in K$ . Finally, recalling that here  $-K_\infty = K_\infty \subset K$ , we see that  $x - [G + G^T]x = -\lambda P_{K_\infty}x \in K$ . ■

**Proposition 9** *Suppose that  $K$  is symmetric. If there exists  $\lambda \geq 0$  such that*

$$(I + \lambda P_{K_\infty})A \in \mathcal{P}_K^{++}$$

*then  $A \in \mathcal{L}_K^{++}$ .*

**Proof:** We choose  $G$  as in Proposition 8 and check that  $\langle A., [G + G^T]. \rangle = \langle [I + \lambda P_{K_\infty}]A., . \rangle$ . By assumption, it results that  $\inf_{x \in K \cap S_1} \langle Ax, [G + G^T]x \rangle > 0$ . The proof is achieved as in Proposition 8. ■

**Proposition 10** *If there exists a symmetric nonsingular  $M$ -matrix  $Q$  such that*

$$QA \in \mathcal{P}_{\mathbb{R}_+^n}$$

*then  $A \in \mathcal{L}_{\mathbb{R}_+^n}$ .*

**Proof:** There exists  $P$  with  $p_{ij} \geq 0, \forall i, j \in \{1, \dots, n\}$  and  $s > \rho(P)$  such that  $Q = sI - P$ . We set  $G := \frac{1}{2s}Q$ . The matrix  $G$  is positive definite and by assumption  $QA \in \mathcal{P}_{\mathbb{R}_+^n}$ . Moreover  $I - (G + G^T) = \frac{1}{s}P$  so that  $[(I - (G + G^T))(\mathbb{R}_+^n) \subset \mathbb{R}_+^n$ . ■

**Proposition 11** *If there exists a symmetric nonsingular  $M$ -matrix  $Q$  such that*

$$QA \in \mathcal{P}_{\mathbb{R}_+^n}^+$$

*then  $A \in \mathcal{L}_{\mathbb{R}_+^n}^{++}$ .*

**Proof:** The proof is similar to the one of Proposition 10. ■



**Proposition 12** *Let us consider a linear state transformation  $z = Lx$ ,  $L \in \mathbb{R}^{n \times n}$  full-rank, and  $LL^T = I$ . Then  $A \in \mathcal{L}_K$  (resp.  $A \in \mathcal{L}_K^{++}$ ) if and only if  $\bar{A} = LAL^T \in \mathcal{L}_{L(K)}$  (resp.  $\bar{A} \in \mathcal{L}_{L(K)}^{++}$ ) with  $G$  (see definition 4 and definition 5) transformed to  $\bar{G} = LGL^T$ .*

**Proof:** The set  $L(K) = \{z \in \mathbb{R}^n : x \in K \text{ and } z = Lx\}$  is convex and closed since  $K_\infty \cap \text{Ker}(L) = \{0\}$  [18, p.71]. Since  $\{z \in L(K)\} \Leftrightarrow \{x \in K\}$ , it follows that  $\{G \in \mathcal{P}_K\} \Leftrightarrow \{\bar{G} \in \mathcal{P}_{L(K)}\}$ . Moreover, for  $x \in \partial K \Leftrightarrow z \in \partial L(K)$  one has  $(I - [G + G^T])x \in K$  if and only if  $(I - [\bar{G} + \bar{G}^T])z \in L(K)$ . ■

**Examples 1** Let us here illustrate the previous results with some simple examples.

i) Let  $K := \mathbb{R}_+ \times \mathbb{R}_+$  and

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

It is clear that  $A \in \mathcal{P}_K$  and from Proposition 2 we obtain that  $A \in \mathcal{L}_K$ .

ii) Let  $K = (\mathbb{R}_+)^n$  and  $A \in \mathbb{R}^{n \times n}$  such that  $A \geq 0$ . Then clearly

$$\langle x, Ax \rangle \geq 0, \forall x \in K$$

and thus  $A \in \mathcal{P}_K$ . Proposition 2 ensures that  $A \in \mathcal{L}_K$ .

iii) Let  $K := \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\}$  and

$$A = \begin{pmatrix} 3 & 0 \\ 1 & -4 \end{pmatrix}.$$

Here  $x \in K \Rightarrow \langle Ax, x \rangle = 3x_1^2$ . Moreover  $K \cap S_1 = \{(x_1, 0) : x_1^2 = 1\}$  and thus  $\inf_{x \in K \cap S_1} \langle Ax, x \rangle = 3$ . It results that  $A \in \mathcal{P}_K^{++}$ . Proposition 3 ensures that  $A \in \mathcal{L}_K^{++}$ .

iv) Let  $K = \mathbb{R}_+ \times \mathbb{R}_+$  and

$$A = \begin{pmatrix} 1 & -10 \\ 0 & 2 \end{pmatrix}.$$

The matrix  $A$  is a nonsingular  $M$ -matrix. Moreover  $K$  is a cone and if  $x \in K$  then  $x_i \bar{e}_i \in K$  ( $i = 1, \dots, n$ ). Using Proposition 4 and remark 7, we obtain that  $A \in \mathcal{L}_K^{++}$ .

v) Let  $K = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$  and

$$A = \begin{pmatrix} 1 & -4 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Setting  $D = \text{diag}\{\frac{1}{5}, 1, \frac{1}{5}\}$ , we check easily that  $\langle Ax, Dx \rangle > 0, \forall x \in K \setminus \{0\}$ . Using Proposition 4, we get  $A \in \mathcal{L}_K^{++}$ .

vi) Let  $K = \{x \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ : \|x\| \leq 1\}$  and

$$A = \begin{pmatrix} 1 & -4 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that if  $x \in \partial K$  then  $x_i \bar{e}_i \in K$  ( $i = 1, \dots, n$ ). Set  $D = \text{diag}\{1, 5, 5\}$ . Here  $\text{tr}\{D\} \geq (n-1) \max_{1 \leq i \leq 3} \{d_{ii}\}$  and thus condition (1) of Proposition 6 holds. Moreover

$$DA = \begin{pmatrix} 1 & -4 & 0 \\ 5 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and thus  $\langle DAx, x \rangle = x_1^2 + 5x_2^2 + x_1x_2 \geq 0, \forall x \in K$ . All the conditions of Proposition 6 hold and thus  $A \in \mathcal{L}_K$ .

vii) Let  $K$  be defined as in example vi) and set

$$A = \begin{pmatrix} 1 & -4 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let  $D$  be defined as in example vi). We have

$$DA = \begin{pmatrix} 1 & -4 & 0 \\ 5 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

and thus  $\langle DAx, x \rangle = x_1^2 + 5x_2^2 + 5x_3^2 + x_1x_2$ . It results that  $\inf_{x \in K \cap S_1} \langle DAx, x \rangle \geq 1$ . From Proposition 7, we deduce that  $A \in \mathcal{L}_K^{++}$ .

viii) Let  $K = \{x \in \mathbb{R}^3 : -1 \leq x_3 \leq 1\}$  and

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix}.$$

The set  $K$  is symmetric and  $K_\infty = \{x \in \mathbb{R}^3 : x_3 = 0\}$ . Then

$$P_{K_\infty} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Setting  $\lambda = 1$ , we see that  $I + \lambda P_{K_\infty} = \text{diag}\{2, 2, 1\}$  and  $\langle [I + \lambda P_{K_\infty}]Ax, x \rangle = 2x_2^2 + 2x_3^2 \geq 0, \forall x \in K$ . Using Proposition 8, we see that  $A \in \mathcal{L}_K$ .

ix) Let  $K$  be defined as in example viii) and set

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -4 & 1 \end{pmatrix}.$$

Setting  $\lambda = 3$ , we see that  $I + \lambda P_{K_\infty} = \text{diag}\{4, 4, 1\}$  and  $([I + \lambda P_{K_\infty}]Ax, x) = 4x_1^2 + 4x_2^2 + x_3^2$ . Thus  $[I + \lambda P_{K_\infty}]A \in \mathcal{P}_K^{++}$  and Proposition 9 ensures that  $A \in \mathcal{L}_K^{++}$ .

x) Let  $K = \{x \in \mathbb{R}^2 : x_1 - 1 \leq x_2 \leq x_1 + 1\}$  and

$$A = \begin{pmatrix} 2 & 2 \\ -2 & 0 \end{pmatrix}.$$

The set  $K$  is symmetric and  $K_\infty = \{x \in \mathbb{R}^2 : x_2 = x_1\}$ . We have

$$P_{K_\infty} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and setting  $\lambda = 2$ , we see that

$$[I + \lambda P_{K_\infty}]A = \begin{pmatrix} 2 & 4 \\ -2 & 2 \end{pmatrix}.$$

Thus  $\langle [I + \lambda P_{K_\infty}]Ax, x \rangle$  is positive definite and  $\inf_{x \in K \cap S_1} (I + \lambda P_{K_\infty})Ax, x > 0$ . Using Proposition 9, we obtain that  $A \in \mathcal{L}_K^{++}$ .

xi) If  $A = \text{diag}\{-a, b\}$  with  $a > 0$ ,  $b > 0$ , then  $A \in \mathcal{P}_K$  for all closed convex sets  $K \subseteq \{-\sqrt{\frac{a}{b}}x_1 \leq x_2 \leq \sqrt{\frac{a}{b}}x_1\}$ .

xii) If

$$A = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix},$$

$b > 0$ ,  $c > 0$ , then **(1)**  $A \in \mathcal{P}_{\mathbb{R}_+^2} \cap \mathcal{P}_{\mathbb{R}_-^2}$  if  $-b + c > 0$ , **(2)**  $A \in \mathcal{P}_{\mathbb{R}_+^2 \times \mathbb{R}_-^2} \cap \mathcal{P}_{\mathbb{R}_-^2 \times \mathbb{R}_+^2}$  if  $-b + c < 0$ . In case **(1)**,  $A \in \mathcal{P}_K^+$  for all closed convex sets  $K \subset \mathbb{R}_+^2 \setminus [\{x_1 = 0\} \cup \{x_2 = 0\}]$  and for all closed convex sets  $K \subset \mathbb{R}_-^2 \setminus [\{x_1 = 0\} \cup \{x_2 = 0\}]$ . In case **(2)** this holds for the quadrants  $\mathbb{R}^- \times \mathbb{R}^+ \setminus [\{x_1 = 0\} \cup \{x_2 = 0\}]$  and  $\mathbb{R}^+ \times \mathbb{R}^- \setminus [\{x_1 = 0\} \cup \{x_2 = 0\}]$ .

## 4 Unstability matrices on a closed convex set

Let  $K \subset \mathbb{R}^n$  be a set satisfying hypothesis  $(H_1) - (H_3)$  (see Section 3).

**Definition 6** A matrix  $A \in \mathbb{R}^{n \times n}$  is said to be Lyapunov unstable on  $K$  if there exists a matrix  $G \in \mathbb{R}^{n \times n}$  and a constant  $\alpha > 0$  such that

- (1)  $G \in \mathcal{P}_K^+$ ,
- (2)  $\langle Ax, [G + G^T]x \rangle \leq -\alpha \langle Gx, x \rangle, \forall x \in K$
- (3)  $x \in \partial K \Rightarrow [G + G^T]x \in K_\infty$ .

The set of Lyapunov unstable matrices on  $K$  will be denoted by  $\mathcal{L}_K^-$ . We have

$$\mathcal{L}_K^- = \{A \in \mathbb{R}^{n \times n} : \exists G \in \mathcal{P}_K^+ \text{ and } \alpha > 0 \text{ such that } -[(G + G^T)A + \alpha G] \in \mathcal{P}_K \\ \text{and } [G + G^T](\partial K) \subset K_\infty\}.$$

**Remark 8** If

$$c := \sup_{x \in K \cap S_1} \langle Ax, [G + G^T]x \rangle < 0$$

then condition (2) of definition 6 holds with  $\alpha = \frac{|c|}{\|G\|}$ .

**Proposition 13** Suppose that  $K$  is a cone. If  $-A \in \mathcal{P}_K^+$ , i.e.

$$\sup_{x \in K \cap S_1} \langle Ax, x \rangle < 0$$

then  $A \in \mathcal{L}_K^-$ .

**Proof:** We set  $G = \frac{1}{2}I$ . The set  $K$  is a cone and thus there exists  $c > 0$  such that  $\langle -Ax, x \rangle \geq c \|x\|^2, \forall x \in K$ . For  $x \in K$ , we have  $\langle Ax, [G + G^T]x \rangle = \langle Ax, x \rangle \leq -c \|x\|^2 = -2c \langle Gx, x \rangle$ . It results that condition (2) of definition 6 holds with  $\alpha = 2c$ . Moreover,  $K_\infty = K$  and thus  $x \in \partial K \subset K \Rightarrow [G + G^T]x = x \in K_\infty$ . ■

**Proposition 14** Suppose that  $K$  is a cone such that

$$x \in \partial K \Rightarrow x_i \bar{e}_i \in K \ (i = 1, \dots, n).$$

If there exists a positive diagonal matrix  $D$  such that

$$-DA \in \mathcal{P}_K^+$$

then  $A \in \mathcal{L}_K^-$ .

**Proof:** We set  $G = \frac{1}{2}D$ . As in the proof of Proposition 4, we check that  $\langle Ax, [G + G^T]x \rangle = \langle DAx, x \rangle$ . Here  $K$  is a cone and thus  $\mathcal{P}_K^+ = \mathcal{P}_K^{++}$ . Thus there exists  $c > 0$  such that  $\langle -DAx, x \rangle \geq c \|x\|^2, \forall x \in K$ . Thus  $\langle Ax, [G + G^T]x \rangle \leq -c \|x\|^2 \leq \frac{-2c}{\|D\|} \langle Gx, x \rangle, \forall x \in K$ . Setting  $\alpha = \frac{2c}{\|D\|}$  we see that condition (2) of Definition 6 holds. Moreover, if  $x \in \partial K$  then  $[G + G^T]x = Dx = \sum_{i=1}^n d_{ii}(x_i \bar{e}_i) \in K_\infty = K$ . ■

**Proposition 15** *If there exist  $\lambda \in \sigma(A) \cap \mathbb{R}$  and  $U_\lambda \in E_{A^T}(\lambda) \setminus \{0\}$  such that*

- (1)  $\lambda < 0$ ,
- (2)  $\ker\{U_\lambda \otimes U_\lambda^T\} \cap K = \{0\}$ ,
- (3)  $x \in \partial K \Rightarrow (U_\lambda \otimes U_\lambda^T)x \in K_\infty$ ,

*then  $A \in \mathcal{L}_K^-$ .*

**Proof:** We first remark that

$$[U_\lambda \otimes U_\lambda^T]x = \langle U_\lambda, x \rangle U_\lambda.$$

Setting  $G := U_\lambda \otimes U_\lambda^T$ , we see that  $G$  is symmetric and

$$\langle Gx, x \rangle = \langle U_\lambda, x \rangle^2 \geq 0, \forall x \in \mathbb{R}^n$$

and thus  $G$  is positive semi-definite. It results that

$$\ker\{G\} = \{x \in \mathbb{R}^n : \langle Gx, x \rangle = 0\}.$$

Assumption (2) ensures that

$$\{x \in K : \langle Gx, x \rangle = 0\} = \{0\}$$

and thus

$$\langle Gx, x \rangle > 0, \forall x \in K \setminus \{0\}.$$

It results that  $G \in \mathcal{P}_K^+$ .

We have  $\langle Ax, [G + G^T]x \rangle = 2\langle Ax, U_\lambda \rangle \langle U_\lambda, x \rangle = 2\langle x, A^T U_\lambda \rangle \langle U_\lambda, x \rangle = 2\lambda \langle U_\lambda, x \rangle^2 = 2\lambda \langle [U_\lambda \otimes U_\lambda^T]x, x \rangle$ . Setting  $\alpha := 2|\lambda|$ , we see that  $\langle Ax, [G + G^T]x \rangle = -\alpha \langle Gx, x \rangle, \forall x \in K$ . This together with assumption (3) yield the conditions of Definition 6.  $\blacksquare$

**Proposition 16** *Suppose that there exist  $\lambda, \mu \in \sigma(A) \cap \mathbb{R}$ ,  $U_\lambda \in E_{A^T}(\lambda) \setminus \{0\}$  and  $U_\mu \in E_{A^T}(\mu) \setminus \{0\}$  such that*

- (1)  $\lambda + \mu < 0$ ,
- (2)  $\langle [U_\lambda \otimes U_\mu^T]x, x \rangle > 0, \forall x \in K \setminus \{0\}$
- (3)  $x \in \partial K \Rightarrow (U_\lambda \otimes U_\mu^T + U_\mu \otimes U_\lambda^T)x \in K_\infty$

*then  $A \in \mathcal{L}_K^-$ .*

**Proof:** We first remark that

$$(U_\lambda \otimes U_\mu^T + U_\mu \otimes U_\lambda^T)x = \langle U_\mu, x \rangle U_\lambda + \langle U_\lambda, x \rangle U_\mu.$$

Let us set

$$G := U_\lambda \otimes U_\mu^T.$$

Assumptions (2) and (3) yield conditions (1) and (3) of Definition 6. We have  $\langle Ax, [G + G^T]x \rangle = \langle Ax, \langle U_\mu, x \rangle U_\lambda + \langle U_\lambda, x \rangle U_\mu \rangle = \langle U_\mu, x \rangle \langle Ax, U_\lambda \rangle + \langle U_\lambda, x \rangle \langle Ax, U_\mu \rangle = \langle U_\mu, x \rangle \langle x, A^T U_\lambda \rangle + \langle U_\lambda, x \rangle \langle x, A^T U_\mu \rangle = (\lambda + \mu) [\langle U_\lambda, x \rangle \langle U_\mu, x \rangle] = (\lambda + \mu) \langle [U_\lambda \otimes U_\mu^T]x, x \rangle$ . and setting  $\alpha := |\lambda + \mu|$ , we obtain that  $\langle Ax, [G + G^T]x \rangle = -\alpha \langle Gx, x \rangle, \forall x \in K$ . ■

**Proposition 17** Suppose that  $K$  is a cone satisfying the properties:

$$\overset{\circ}{K} \neq \emptyset \tag{10}$$

$$x \in K, y \in K \Rightarrow \langle x, y \rangle \geq 0 \tag{11}$$

$$x \in \overset{\circ}{K}, y \in K, \langle x, y \rangle = 0 \Rightarrow y = 0 \tag{12}$$

If there exist  $\lambda < 0$  and  $U_\lambda \in \overset{\circ}{K}$  such that

$$A^T U_\lambda - \lambda U_\lambda \in -N_K(0)$$

then  $A \in \mathcal{L}_K^-$ .

**Proof:** We set  $G := U_\lambda \otimes U_\lambda^T$ . Note that (10) together with (12) ensure that  $U_\lambda \neq 0$ . We have

$$\langle Gx, x \rangle = \langle U_\lambda, x \rangle^2 \geq 0, \forall x \in \mathbb{R}^n.$$

We remark now that

$$\langle Gx, x \rangle > 0, \forall x \in K \setminus \{0\}.$$

Indeed, suppose by contradiction that  $\langle Gx, x \rangle = 0$  for some  $x \in K \setminus \{0\}$ . Then  $\langle U_\lambda, x \rangle = 0$  and since  $U_\lambda \in \overset{\circ}{K}$  we obtain from (12) that  $x = 0$  which is a contradiction. Proposition 13 ensures that  $G \in \mathcal{P}_K^{++}$  since  $K$  is a cone. We see also that if  $x \in \partial K \subset K$  then  $(G + G^T)x = 2\langle U_\lambda, x \rangle U_\lambda \in K = K_\infty$ . Indeed  $U_\lambda \in K, \langle U_\lambda, x \rangle \geq 0$  because of (10) and  $K$  is a cone. Finally,  $\langle Ax, [G + G^T]x \rangle = 2\langle x, A^T U_\lambda \rangle \langle U_\lambda, x \rangle$  and thus if  $x \in K$  then  $\langle U_\lambda, x \rangle \geq 0$  and  $\langle x, A^T U_\lambda \rangle \geq \lambda \langle x, U_\lambda \rangle$ . It results that  $\langle Ax, [G + G^T]x \rangle \geq 2\lambda \langle U_\lambda, x \rangle^2 = 2\lambda \langle Gx, x \rangle$ . ■

**Examples 2** i) Let  $K = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$  and

$$A = \begin{pmatrix} -1 & -3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We see easily that  $-A \in \mathcal{P}_K^+$  and thus from Proposition 13, we deduce that  $A \in \mathcal{L}_K^-$ .

ii) Let  $K = \{x \in \mathbb{R}^2 : -\frac{1}{2}x_1 \leq x_2 \leq -x_1\}$  and

$$A = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}.$$

Here  $\langle Ax, x \rangle = x_1 x_2$  and if  $(x_1, x_2) \in K \setminus \{0\}$  then  $x_1 < 0$  and  $x_2 > 0$ . It results that  $\langle Ax, x \rangle < 0, \forall x \in K \setminus \{0\}$ . Proposition 13 ensures that  $A \in \mathcal{L}_K^-$ .

iii) Let  $K = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$  and

$$A = \begin{pmatrix} -1 & 4 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Setting  $D = \text{diag}\{1, 5, 5\}$ , we see that  $-DA \in \mathcal{P}_K^+$  (see example 1 vii)) and Proposition 14 ensures that  $A \in \mathcal{L}_K^-$ .

iv) Let  $K = \mathbb{R}_+ \times \mathbb{R}_+$  and

$$A = \begin{pmatrix} -1 & -3 \\ 0 & 2 \end{pmatrix}.$$

Then  $\sigma(A) = \{-1, 2\}$  and let  $\lambda = -1$ , we check that  $U_\lambda = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in E_{A^T}(-1)$ . Here

$$U_\lambda \otimes U_\lambda^T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and thus  $x \in \partial K \subset K \Rightarrow [U_\lambda \otimes U_\lambda^T]x \in K_\infty = K$ . Moreover  $\ker\{U_\lambda \otimes U_\lambda^T\} = \{x \in \mathbb{R}^2 : x_2 = -x_1\}$  so that  $K \cap \ker\{U_\lambda \otimes U_\lambda^T\} = \{0\}$ . Using Proposition 15, we obtain that  $A \in \mathcal{L}_K^-$ .

Note that the result can also be obtained from Proposition 17 since  $U_\lambda \in \overset{\circ}{K}$ ,  $A^T U_\lambda - \lambda U_\lambda = 0 \in -N_K(0)$  and  $K$  satisfies de conditions 10 – 12 required in Proposition 17.

v) Let  $K = \mathbb{R}_+ \times \mathbb{R}_+$  and

$$A = \begin{pmatrix} -6 & -6 \\ 2 & 2 \end{pmatrix}.$$

Then  $\sigma(A) = \{-4, 0\}$ . Set  $\lambda = -4$  and  $\mu = 0$ . We see that  $U_\lambda = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in E_{A^T}(-4)$  and

$U_\mu = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \in E_{A^T}(0)$ . Here

$$U_\lambda \otimes U_\mu^T = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$$

and thus  $\langle [U_\lambda \otimes U_\mu^T]x, x \rangle = x_1^2 + 3x_2^2 + 4x_1x_2 > 0, \forall x \in K \setminus \{0\}$ . Moreover if  $x \in \partial K \subset K$  then

$$(U_\lambda \otimes U_\mu^T + U_\mu \otimes U_\lambda^T)x = \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in K_\infty = K.$$

## 5 Stability of Linear Evolution Variational Inequalities

Let  $K \subset \mathbb{R}^n$  be a closed convex set such that  $0 \in K$ . We consider Problem  $P(t_0, x_0)$  with  $F \equiv 0$ , i.e. : Find  $x \in C^0([t_0, \infty); \mathbb{R}^n)$  such that  $\frac{dx}{dt} \in L_{\text{loc}}^\infty(t_0, +\infty; \mathbb{R}^n)$  and

$$\langle \frac{dx}{dt}(t) + Ax(t), v - x(t) \rangle \geq 0, \forall v \in K, \text{ a.e. } t \geq t_0 \quad (13)$$

$$x(t) \in K, t \geq t_0 \quad (14)$$

$$x(t_0) = x_0 \quad (15)$$

**Theorem 5** *Let  $K \subset \mathbb{R}^n$  be a set satisfying hypothesis  $(H_1)$ -( $H_3$ ).*

- *i) If  $A \in \mathcal{L}_K$  then the trivial solution of (13)-(14) is stable.*
- *ii) If  $A \in \mathcal{L}_K^{++}$  then the trivial solution of (13)-(14) is asymptotically stable.*
- *iii) Suppose here in addition that  $K \setminus \{0\} \neq \emptyset$ . If  $A \in \mathcal{L}_K^-$  then the trivial solution of (13)-(14) is unstable.*

**Proof:** i)  $A \in \mathcal{L}_K$  and thus there exists a matrix  $G \in \mathbb{R}^{n \times n}$  such that

$$\inf_{x \in K \cap S_1} \langle Gx, x \rangle > 0 \quad (16)$$

$$\langle Ax, [G + G^T]x \rangle \geq 0, \forall x \in K \quad (17)$$

and

$$x \in \partial K \Rightarrow [I - [G + G^T]]x \in K \quad (18)$$

Let  $V \in C^1(\mathbb{R}^n; \mathbb{R})$  be defined by

$$V(x) = \frac{1}{2} \langle [G + G^T]x, x \rangle \quad (19)$$

Then  $V'(x) = [G + G^T]x$  and we see that all the assumptions of Theorem 2 are satisfied. Indeed (16) ensures the existence of a constant  $k > 0$  (see Remark 3) such that

$$V(x) \geq k \|x\|^2, \forall x \in K.$$



It is clear that  $V(0) = 0$ . Finally from (17) and (18) we deduce that

$$\langle Ax, V'(x) \rangle \geq 0, \forall x \in K$$

and

$$x \in \partial K \Rightarrow x - V'(x) \in K.$$

The conclusion follows from Theorem 2.

ii)  $A \in \mathcal{L}_K^{++}$  and thus there exists a matrix  $G \in \mathbb{R}^{n \times n}$  satisfying (16), (18) and

$$\inf_{x \in K \cap S_1} \langle Ax, [G + G^T]x \rangle > 0 \quad (20)$$

We define  $V \in C^1(\mathbb{R}^n; \mathbb{R})$  as in (19) and we verify as in Part i) that assumptions (1), (2) and (3) of Theorem 3 are satisfied. Moreover, from (20), we deduce the existence of a constant  $c > 0$  such that

$$\langle Ax, [G + G^T]x \rangle \geq c \|x\|^2, \forall x \in K.$$

It results that

$$\langle Ax, [G + G^T]x \rangle \geq \frac{c}{\|G + G^T\|} \langle [G + G^T]x, x \rangle, \forall x \in K.$$

Thus

$$\langle Ax, [G + G^T]x \rangle \geq \lambda V(x), \forall x \in K$$

with  $\lambda := \frac{2c}{\|G + G^T\|}$ . This yields assumption (4) of Theorem 3. The conclusion follows from Theorem 3.

iii) If  $A \in \mathcal{L}_K^-$  then there exists a matrix  $G \in \mathbb{R}^{n \times n}$  and  $\alpha > 0$  such that

$$\langle Gx, x \rangle > 0, \forall x \in K \setminus \{0\} \quad (21)$$

$$x \in \partial K \Rightarrow [G + G^T]x \in K_\infty \quad (22)$$

and

$$(Ax, [G + G^T]x) \leq -\alpha \langle Gx, x \rangle, \forall x \in K \quad (23)$$

Let  $V \in C^1(\mathbb{R}^n; \mathbb{R})$  be defined by

$$V(x) = \frac{1}{2} \langle [G + G^T]x, x \rangle$$

We see that all the conditions of Theorem 4 are satisfied. It is clear that assumption (1) is satisfied with  $b(t) = \frac{1}{2} \|G + G^T\| t^2$ . Condition (21) ensures that assumption (2) is satisfied. Condition (22) yields assumption (3). Finally, from (23), we deduce that

$$\langle Ax, V'(x) \rangle \leq -\alpha V(x), \forall x \in K,$$

so that assumption (4) holds too. The conclusion follows from Theorem 4. ■

**Example 1** Let us here use Theorem 5 to discuss the stability of the trivial solution of Problem (13)-(14) for different matrices  $A \in \mathbb{R}^{n \times n}$  and sets  $K \subset \mathbb{R}^n$  given in Examples 1 and Examples 2. We also discuss the stability of the trivial solution of Problem (13)-(14) without constraint, i.e. with  $K = \mathbb{R}^n$ , and see by the way that the stability of an equilibrium may substantially change as soon as inequality constraints are involved.

Matrix $A$ and set $K$ given in	$\sigma(A)$	Trivial solution of $\frac{du}{dt} + Au = 0$	Trivial solution of $(\frac{du}{dt} + Au, v - u) \geq 0, \forall v \in K$
Example 1 i)	$\{1 - \sqrt{2}, 1 + \sqrt{2}\}$	Unstable	Asymptotically stable
Example 1 iii)	$\{-4, 3\}$	Unstable	Asymptotically stable
Example 1 iv)	$\{1, 2\}$	Asymptotically stable	Asymptotically stable
Example 1 v)	$\{0.8769 - 2.0113i, 0.8769 + 2.0113i, 1.2463\}$	Asymptotically stable	Asymptotically stable
Example 1 vi)	$\{0, 1 - 2i, 1 + 2i\}$	Stable	Stable
Example 1 vii)	$\{1, 1 - 2i, 1 + 2i\}$	Asymptotically stable	Asymptotically stable
Example 1 viii)	$\{0, \frac{3-\sqrt{7}i}{2}, \frac{3+\sqrt{7}i}{2}\}$	Stable	Stable
Example 1 ix)	$\{1, 1 - 2i, 1 + 2i\}$	Asymptotically stable	Asymptotically stable
Example 1 x)	$\{1 - \sqrt{3}i, 1 + \sqrt{3}i\}$	Asymptotically stable	Asymptotically stable
Example 2 i)	$\{-1\}$	Unstable	Unstable
Example 2 ii)	$\{-\sqrt{2}i, \sqrt{2}i\}$	Stable	Unstable
Example 2 iii)	$\{-1, -1 - 2i, -1 + 2i\}$	Unstable	Unstable
Example 2 iv)	$\{-1, 2\}$	Unstable	Unstable
Example 2 v)	$\{-4, 0\}$	Unstable	Unstable

**Remark 9** Note that if  $K \subset \mathbb{R}^n$  is a subspace then the solution  $x$  of Problem (13) – (15) satisfies the system

$$x(t) = P_K(x(t) - x'(t) - Ax(t)).$$

Here  $P_K$  is linear,  $x(t), x'(t) \in K$  and thus

$$\frac{dx}{dt}(t) + P_K Ax(t) = 0, \text{ a.e. } t \geq t_0,$$

It results that

$$x(t) = e^{-P_K A t} x_0$$

with

$$e^{-P_K A t} = \frac{1}{2\pi i} \oint_{\Gamma} e^{-tz} (zI - P_K A)^{-1} dz$$

where  $\Gamma$  denotes a Jordan curve enclosing an open disk containing  $\sigma(P_K A)$ . It is known that variational inequalities, complementarity and projected dynamical systems, are closely related one to each other [34].

## 6 Nonlinear perturbations of linear variational inequalities

Let  $K \subset \mathbb{R}^n$  be a closed convex set such that  $0 \in K$ . Let us now consider Problem  $P(t_0, x_0)$  : Find  $x \in C^0([t_0, \infty); \mathbb{R}^n)$  such that  $\frac{dx}{dt} \in L_{\text{loc}}^\infty(t_0, +\infty; \mathbb{R}^n)$  and

$$\left\langle \frac{dx}{dt}(t) + Ax(t) + F(x(t)), v - x(t) \right\rangle \geq 0, \forall v \in K, \text{ a.e. } t \geq t_0 \quad (24)$$

$$x(t) \in K, t \geq t_0 \quad (25)$$

$$x(t_0) = x_0 \quad (26)$$

**Theorem 6** Let  $K \subset \mathbb{R}^n$  be a set satisfying hypothesis  $(H_1)$ – $(H_3)$ . Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be given by

$$F = F_1 + \Phi'$$

where  $F_1$  is Lipschitz continuous and  $\Phi \in C^1(\mathbb{R}^n; \mathbb{R})$  is convex. Suppose also that

$$\lim_{\|x\| \rightarrow 0} \frac{\|F(x)\|}{\|x\|} = 0 \quad (27)$$

If  $A \in \mathcal{L}_K^{++}$  then the trivial solution of (24)–(25) is asymptotically stable.

**Proof:** There exists a matrix  $G \in \mathbb{R}^{n \times n}$  such that

$$\inf_{x \in K \cap S_1} \langle Gx, x \rangle > 0 \quad (28)$$

$$\inf_{x \in K \cap S_1} \langle Ax, [G + G^T]x \rangle > 0 \quad (29)$$

and

$$x \in \partial K \Rightarrow x - [G + G^T]x \in K \quad (30)$$

Our aim is to verify that all conditions of Theorem 3 are satisfied with  $V \in C^1(\mathbb{R}^n; \mathbb{R})$  defined by

$$V(x) = \frac{1}{2} \langle [G + G^T]x, x \rangle$$

From (28), we see that there exists a constant  $c_1 > 0$  such that

$$V(x) = \langle Gx, x \rangle \geq c_1 \|x\|^2, \forall x \in K.$$

This yields assumption (1) of Theorem 3. It is clear that  $V(0) = 0$  so that assumption (2) of Theorem 3 is also satisfied. Here  $V'(x) = [G + G^T]x$  and (30) yields assumption (3) of Theorem 3. Finally, from (29) we obtain that

$$\langle Ax, V'(x) \rangle \geq c_2 \|x\|^2, \forall x \in K,$$

for some constant  $c_2 > 0$ . On the other hand, because of (27) there exists a constant  $\sigma > 0$  such that

$$\|x\| \leq \sigma \Rightarrow \|F(x)\| \leq \frac{c_2}{\|G + G^T\|} \|x\|.$$

Thus, if  $\|x\| \leq \sigma$  then

$$\begin{aligned} \langle F(x), V'(x) \rangle &= \frac{1}{2} \langle F(x), [G + G^T]x \rangle \\ &\geq -\frac{1}{2} \|G + G^T\| \|F(x)\| \|x\| \geq -\frac{1}{2} c_2 \|x\|^2 \end{aligned}$$

It results that

$$\langle Ax + F(x), V'(x) \rangle \geq \frac{c_2}{2} \|x\|^2, \forall x \in K, \|x\| \leq \sigma$$

and thus assumption (4) of Theorem 3 holds with  $\lambda = \frac{c_2}{\|G + G^T\|}$ . ■

**Theorem 7** Let  $K \subset \mathbb{R}^n$  be a set satisfying hypothesis  $(H_1)$ – $(H_3)$ . Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be given by

$$F = F_1 + \Phi'$$

where  $F_1$  is Lipschitz continuous and  $\Phi \in C^1(\mathbb{R}^n; \mathbb{R})$  is convex. Suppose also that

$$-\Phi'(0) \in N_K(0) \tag{31}$$

and

$$\lim_{\|x\| \rightarrow 0} \frac{\|F_1(x)\|}{\|x\|} = 0 \tag{32}$$

If  $A \in \mathcal{P}_K^{++}$  then the trivial solution of (24)–(25) is asymptotically stable.

**Proof:** We first note that (31) and (32) ensure that  $-F(0) \in N_K(0)$ . Let us now define  $V \in C^1(\mathbb{R}^n; \mathbb{R})$  by the formula

$$V(x) = \frac{1}{2} \|x\|^2.$$

We see that all the assumptions of Theorem 3 are satisfied. Indeed, assumptions (1), (2) and (3) are clearly satisfied. Moreover  $A \in \mathcal{P}_K^{++}$  and thus there exists a constant  $c > 0$  such that

$$\langle Ax, x \rangle \geq c \|x\|^2, \forall x \in K.$$

On the other hand  $\Phi'$  is monotone and thus

$$\langle \Phi'(x), x \rangle \geq \langle \Phi'(0), x \rangle, \forall x \in \mathbb{R}^n.$$

From (31) it results that

$$\langle \Phi'(x), x \rangle \geq 0, \forall x \in K.$$

Finally, condition (32) ensures the existence of  $\sigma > 0$  such that

$$\|x\| \leq \sigma \Rightarrow \|F_1(x)\| \leq \frac{c}{2} \|x\|.$$

It results that if  $x \in K$  then

$$\begin{aligned} \langle Ax + F_1(x) + \Psi'(x), V'(x) \rangle &= \langle Ax + F_1(x) + \Phi'(x), x \rangle \\ &\geq c \|x\|^2 - \|F_1(x)\| \|x\| \end{aligned}$$

and thus

$$\langle Ax + F_1(x) + \Phi'(x), V'(x) \rangle \geq \frac{c}{2} \|x\|^2, \forall x \in K, \|x\| \leq \sigma.$$

That means that assumption (4) of Theorem 3 is satisfied with  $\lambda = c$ . The conclusion follows from Theorem 3. ■

## 7 Applications

Let us here discuss the stability of a system described by a transfer function  $G(s) = C(sI - A)^{-1}B$  and a feedback branch containing a sector static nonlinearity, known as the absolute stability problem [14] [28] (see figure 1). Here  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  and  $C \in \mathbb{R}^{m \times n}$ . The static nonlinearity  $y_L = \phi(y)$  is usually assumed to be a locally Lipschitz single-valued function [28, §10.1], possibly time-varying and piecewise continuous in  $t$ .

The feedback nonlinearity is here assumed to be a multivalued monotone mapping of the form  $\partial\psi_K$  where  $K \subset \mathbb{R}^n$  is a set satisfying conditions (H<sub>1</sub>)-(H<sub>3</sub>) of Section 3,  $\psi_K$  is the indicator function of  $K$  and  $\partial$  denotes the convex subdifferential operator. The study of such systems has been initiated in [3]. The state space equations of such a system are given by: Find  $x \in C^0([0, \infty); \mathbb{R}^n)$  such that  $\frac{dx}{dt} \in L_{\text{loc}}^\infty(0, +\infty; \mathbb{R}^n)$  and

$$\frac{dx}{dt}(t) = Ax(t) - By_L(t), \text{ a.e. } t \geq 0 \quad (33)$$

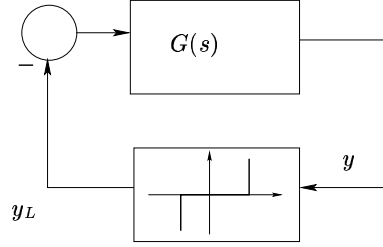


Figure 1: The absolute stability problem.

$$y(t) = Cx(t), t \geq 0 \quad (34)$$

$$y(t) \in K, t \geq 0 \quad (35)$$

$$y_L(t) \in \partial\psi_K(y(t)), t \geq 0 \quad (36)$$

$$x(0) = x_0 \quad (37)$$

Assume there exists a symmetric and invertible matrix  $R \in \mathbb{R}^{n \times n}$  such that  $R^{-2}C^T = B$ . Suppose also that there exists

$$y_0 := CR^{-1}x_0 \in \text{int}\{K\}. \quad (38)$$

Then using the change of state vector  $z = Rx$  and setting

$$\bar{K} = \{h \in \mathbb{R}^n : CR^{-1}h \in K\} \quad (39)$$

we see that problem (33)-(37) is equivalent to the following one: find  $z \in C^0([0, \infty); \mathbb{R}^n)$  such that  $\frac{dz}{dt} \in L_{\text{loc}}^\infty([0, \infty); \mathbb{R}^n)$  and

$$\left\langle \frac{dz}{dt}(t) - RAR^{-1}z(t), v - z(t) \right\rangle \geq 0, \forall v \in \bar{K}, \text{ a.e. } t \geq 0$$

$$z(t) \in \bar{K}, t \geq 0,$$

$$z(0) = Rx_0.$$

Indeed, it suffices to remark that

$$Cx \in K \Leftrightarrow z \in \bar{K},$$

$$x(0) = x_0 \Leftrightarrow z(0) = Rx_0$$

and

$$\begin{aligned} \frac{dx}{dt} \in Ax - B\partial\psi_K(Cx) &\Leftrightarrow R\frac{dx}{dt} \in RAR^{-1}Rx - RB\partial\psi_K(CR^{-1}Rx) \\ \Leftrightarrow \frac{dz}{dt} \in RAR^{-1}z - R^{-1}R^2B\partial\psi_K(CR^{-1}z) &\Leftrightarrow \frac{dz}{dt} \in RAR^{-1}z - R^{-1}C^T\partial\psi_K(CR^{-1}z) \\ &\Leftrightarrow \frac{dz}{dt} \in RAR^{-1}z - \partial\psi_{\bar{K}}(z). \end{aligned}$$

Indeed,  $\psi_{\bar{K}}(z) = (\psi_K \circ CR^{-1})(z)$  and thanks to (38) we obtain  $\partial\psi_{\bar{K}}(z) = R^{-1}C^T\partial\psi_K(CR^{-1}z)$ . We remark also that the set  $\bar{K}$  satisfies the conditions (H<sub>1</sub>)-(H<sub>3</sub>).

Applying now the results of the previous sections, we may discuss the stability of the trivial solution of problem (33)-(36).

**Corollary 1** *Let  $K \subset \mathbb{R}^n$  be a set satisfying hypothesis (H<sub>1</sub>)-(H<sub>3</sub>) together with condition (38) and define  $\bar{K}$  as in (39). Suppose that there exists a symmetric and invertible matrix  $R \in \mathbb{R}^{n \times n}$  such that  $R^{-2}C^T = B$ .*

- i) *If  $-RAR^{-1} \in \mathcal{L}_{\bar{K}}$  then the trivial equilibrium point of (33)-(36) is stable.*
- ii) *If  $-RAR^{-1} \in \mathcal{L}_{\bar{K}}^{++}$  then the trivial equilibrium point of (33)-(36) is asymptotically stable.*
- iii) *Suppose here in addition that  $K \setminus \{0\} \neq \emptyset$ . If  $-RAR^{-1} \in \mathcal{L}_{\bar{K}}^-$  then the trivial equilibrium point of (33)-(36) is unstable.*

**Example 2** Assume that  $G(s) = C(sI - A)^{-1}B$ , with  $(A, B, C)$  a minimal representation, is strictly positive real (SPR), i.e.  $\operatorname{re}\{G(iw)\} > 0, \forall w \in \mathbb{R}$ . From the Kalman-Yakubovitch-Popov Lemma there exist  $G = G^T$  positive definite and  $Q = Q^T$  positive definite such that  $GA + A^TG = -Q$  and  $B^TG = C$ . Choosing  $R$  as the symmetric square root of  $G$ , i.e.  $R = R^T$ ,  $R$  positive definite and  $R^2 = G$ , we see that  $B^TR^2 = C$  and thus  $R^{-2}C^T = B$ . Moreover

$$\langle GAx, x \rangle + \langle A^TGx, x \rangle = -\langle Qx, x \rangle, \forall x \in \mathbb{R}^n. \quad (40)$$

Thus

$$\langle Ax, Gx \rangle = -\frac{1}{2}\langle Qx, x \rangle, \forall x \in \mathbb{R}^n. \quad (41)$$

It results that

$$-\langle RAx, Rx \rangle > 0, \forall x \in \mathbb{R}^n \setminus \{0\}. \quad (42)$$

Setting  $z = Rx$ , we see that

$$-\langle RAR^{-1}z, z \rangle > 0, \forall z \in \mathbb{R}^n \setminus \{0\}. \quad (43)$$

So  $-RAR^{-1} \in \mathcal{P}_{\mathbb{R}^n}^{++} \subset \mathcal{P}_{\bar{K}}^{++} \subset \mathcal{L}_{\bar{K}}^{++}$ . All the conditions of Corollary 1 (part ii)) are satisfied and the trivial solution of (33)-(36) is asymptotically stable. The results of [3] with  $\varphi = \Psi_K$  and [16, theorem 11.2] are here recovered. In case  $G(s)$  is positive real (PR) then corollary 1 (part i)) applies. As shown in [3] the equilibrium point is unique in this case.

In relationship with proposition 12 we have the following:

**Corollary 2** *Consider a dissipative linear complementarity system  $\dot{x} = Ax + B\lambda$ ,  $0 \leq y = Cx \perp \lambda \geq 0$ , with  $(A, B, C)$  a PR (resp. SPR) transfer function. Then proposition 2 (resp. proposition 3) applies whatever the state space representation  $w = Lx$  with  $L \in \mathbb{R}^{n \times n}$ ,  $L$  nonsingular and  $LL^T = I$ .*

**Proof:** The proof follows from the calculations in (40)-(43) (or using the results in [3] or [16]). Indeed whatever transformation  $w = Lx$ , the above shows that there is always a transformation  $z = Rw = RLx$  such that the transformed evolution matrix satisfies proposition 2. ■

**Remark 10** For a relative degree 0 passive LCS ( $y = Cx + D\lambda$ ,  $D + D^T$  positive definite) the framework in this paper no longer applies since the system is an ordinary differential equation (with discontinuous but single-valued right-hand-side) and no longer an inclusion. This is easily seen since  $\lambda$  is the unique solution of a linear complementarity problem with matrix  $\frac{D+D^T}{2}$  [20].

Electrical circuits with ideal diodes and relative degree one between  $y$  and  $\lambda$  (see (3)) are an example of dissipative systems that fit within this framework with PR or SPR transfer functions  $G(s)$  in feedback connection with the corner law [21] [23]. Let us notice that passive LCS have the operator  $\lambda \mapsto y$  which is dissipative, but not necessarily not the operator  $u \mapsto y$ . Hence passive LCS may not be asymptotically stabilised by output feedback  $u = -ky$ ,  $k > 0$ , as passive unconstrained systems are [14].

## 8 Conclusions

In this paper the stability of linear evolution variational inequalities is studied. These dynamical systems have unilateral effects, hence are nonsmooth and nonlinear. The Lyapunov stability is considered, and Lyapunov's second method is investigated. It is shown that the extension is nontrivial, and that the stability of the unconstrained system may drastically differ from that of the constrained system. Examples are given to illustrate the developments, as well as criteria which allow one to test the stability (or the instability). Links to other classes of hybrid dynamical systems (like differential inclusions, complementarity systems) are provided.

## References

- [1] A. Berman, M. Neumann, R.J. Stern, 1989 *Nonnegative Matrices in Dynamic Systems*, John Wiley and Sons.
- [2] F. Blanchini, 1999 "Set invariance in control", *Automatica*, vol.35, no 11, pp.1747-1767.



- [3] B. Brogliato, 2002 “Absolute stability and the Lagrange-Dirichlet theorem with monotone multivalued mappings”, submitted.
- [4] B. Brogliato, 2001 “On the control of nonsmooth complementarity dynamical systems”, *Philosophical Transactions of the Royal Society, series A*, vol.359, no 1789, pp.2369-2384.
- [5] B. Brogliato, 2003 “Some perspectives on the analysis and control of complementarity systems”, to appear in *IEEE Transactions on Automatic Control*, vol.48, no 6, June.
- [6] D. Goeleven, M. Motreanu, V. Motreanu, 2002 “On the stability of stationary solutions of evolution variational inequalities”, to appear in *Advances in Nonlinear Variational Inequalities*, 2002.
- [7] J.P. Hespanha, A.S. Morse, 1999 “Stabilization of nonholonomic integrators via logic-based switching”, *Automatica*, vol.35, no 3, pp.385-394.
- [8] J.A. Ball, M.V. Day, T. Yu, P. Kachroo, 1999 “Robust  $L_2$ -gain control for nonlinear systems with projection dynamics and input constraints: an example from traffic control”, *Automatica*, vol.35, no 3, pp.429-444.
- [9] P. Quittner, 1989 “On the principle of linearized stability for variational inequalities”, *Math. Ann.*, 283, 257-270.
- [10] P. Quittner, 1990 “On the stability of stationary solutions of parabolic variational inequalities”, *Czech. Math. Journal*, 40, 472-474.
- [11] P. Quittner, 1990 “An instability criterion for variational inequalities”, *Nonlinear Analysis TMA*, 15, 1167-1180.
- [12] N. Rouche and J. Mawhin, 1973 *Equations Différentielles Ordinaires*, Tome 2, Masson and Cie, Paris.
- [13] N. Rouche, P. Habets, M. Laloy, 1977 *Stability Theory by Liapunov's Direct Method*, Springer Verlag, New York.
- [14] R. Lozano, B. Brogliato, O. Egeland, B. Maschke, 2000 *Dissipative Systems Analysis and Control*, Springer, London, CCES.
- [15] A. van der Schaft, H. Schumacher, 2000 *An Introduction to Hybrid Dynamical Systems*, Springer, London, LNCIS 251.
- [16] K. Camlibel, W.P.M.H. Heemels, H. Schumacher, 2002 “On linear passive complementarity systems”, *European Journal of Control*, vol.8, no 3, pp.220-237.
- [17] W.P.M.H. Heemels, S. Weiland, 2000 “Linear complementarity systems”, *SIAM J. Applied Math.*, vol.60, no 4, pp.1234-1269.

- [18] J.B. Hiriart-Urruty, C. Lemaréchal, 2001 *Fundamentals of Convex Analysis*, Springer Grundlehren Text Editions, Berlin Heidelberg.
- [19] D. Goeleven, D. Motreanu, Y. Dumont, M. Rochdi, 2002 *Variational and Hemivariational Inequalities*, Kluwer Academic Publishers, Boston.
- [20] K.G. Murty, 1997 *Linear Complementarity, Linear and Nonlinear Programming*, available at <http://www-personal.engin.umich.edu/~murty/book/LCPbook/>
- [21] B. Brogliato, 1999 *Nonsmooth Mechanics*, Springer CCES, 2nd edition, London. Erratum and addenda available at <http://www.inrialpes.fr/bip/people/brogliato/brogli.html>
- [22] M. Frémond, 2002 *Non-Smooth Thermomechanics*, Springer Verlag, Berlin-Heidelberg.
- [23] Ch. Glocker, 2001 *Set Valued Force Laws*, Springer, Lecture Notes in Applied Mechanics, LNAM 1.
- [24] R.A. Horn and C.R. Johnson, 1994 *Topics in Matrix Analysis*, Cambridge University Press.
- [25] M. Kunze, M. Monteiro Marques, 2000 "An introduction to Moreau's sweeping process", in *Impacts in Mechanical Systems*, Springer Lecture Notes in Physics LNP 551, B. Brogliato (Ed.), pp.1-60.
- [26] A.A. ten Dam, E. Dwarshuis, J.C. Willems, 1997 "The contact problem for linear continuous-time dynamical systems: a geometric approach", IEEE Transactions on Automatic Control, vol.42, no 4, pp.458-472, April.
- [27] M. Vassilaki, G. Bitsoris, 1989, "Constrained regulation of linear continuous-time dynamical systems", Systems and Control Letters, vol.13, pp.247-252.
- [28] H.K. Khalil, 1996 *Nonlinear Systems*, 2nd edition, Prentice-Hall.
- [29] D. Goeleven, G.E. Stravoulakis, G. Salmon, P.D. Panagiotopoulos, 1997 "Solvability theory and projection methods for a class of singular variational inequalities: Elastostatic unilateral contact applications", J. Optimization Theory and Applications, vol.95, no 2, pp.263-293.
- [30] J.C. Yao, 1994 "Variational inequalities with generalized monotone operators", Mathematics of Operations Research, vol.19, no 3, pp.691-705.
- [31] K.H. Johansson, A. Rantzer, K.J. Astrom, 1999 "Fast switches in relay feedback systems", Automatica, vol.35, no 4, pp.539-552.
- [32] A.N. Michel, B. Hu, 1999 "Towards a stability theory of general hybrid systems", Automatica, vol.35, no 3, pp.371-384.

- [33] A.N. Michel, K. Wang, B. Hu, 2001 *Qualitative Theory of Dynamical Systems*, 2nd Edition, Marcel Dekker, Pure and Applied Mathematics, New York.
- [34] W.P.M.H. Heemels, J.M. Schumacher, S. Weiland, 2000 “Projected dynamical systems in a complementary formalism”, *Operations Research Letters*, vol.27, pp.83-91.
- [35] P. Dupuis, A. Nagurney, 1993 “Dynamical systems and variational inequalities”, *Ann. Oper. Res.*, vol.44, pp.9-42.
- [36] A. Nagurney, D. Zhang “Projected dynamical systems and variational inequalities with applications”, *International Series in Operations Research and Management Sciences*, 2, Kluwer Academic Publishers; ISBN: 0792396375; (January 1996).
- [37] D. Zhang, A. Nagurney, 1995 “On the Stability of Projected Dynamical Systems”, *Journal of Optimization Theory and Applications*, vol.85, no 1, pp.97-124.

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